Multisensor Constrained Estimation with Unscented Transformation

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Abstract – *The paper focuses on multisensor constrained estimation. The possibility of an extension of a current unconstrained fusion technique is discussed for equality constrained estimates. Basic approaches to multisensor constrained estimation with fusion centre are proposed and discussed. The stress is laid on the constraint in or outside the estimation loop and on the constrained fusion of unconstrained or constrained estimates. The constraint is applied by the estimate projection method with the use of the unscented transformation which does not require the computation of derivatives and detour the problem of singular covariance matrices that are obtained by linearising the projection. A ground tracking example is given and the approaches are compared.*

Keywords: Multisensor fusion, filtering, constrained estimation, unscented transformation, nonlinear constraint

1 Introduction

In estimation theory, the system dynamics is modelled by relations with a deterministic part, that represents known physical laws, and a stochastic part, which may express that the physical laws are approximative or that some unknown disturbances affect the system. The classical approach assumes that the system is described by the stochastic model exactly, i.e. there is no distinction between the system and its model. However, there can be other physical laws whose incorporation into the model dynamics is complicated. These laws explicitly say that the model does not match the system and that the state obeys some constraints.

The classical estimation methods can be found in [1], this book contains various Kalman filter generalisations and numerically stable algorihms. A survey of constrained estimation approaches is made in [2]. The constraints can be enforced by modifying the model dynamics, by using them as a perfect pseudo measurements or by modifying the estimates. The ground tracking is a typical example of the constrained estimation problem.

Nonlinear models and/or nonlinear constraints require computationally demanding approaches like Monte-Carlo algorithms or some less or more crude approximation. In the latter cases, a derivative of a nonlinear function has to be computed usually, however, a derivative-free approaches [3] can detour this problem. A nonlinear global filter with nonlinear constraints was proposed in [4].

The multisensor estimation assumes that the system state is estimated by multiple estimators whose state estimates are fused. Practical aspects of the fusion are shown in [5]. Several fusion algorithms exist, however, they were designed in the classical unconstrained framework where the common information can be computed or eliminated. The difference between the model and the system limits the use of the algorithms.

The goal of the paper is to propose basic approaches to multisensor constrained estimation. To point out some difficulties in the fusion of equality constrained estimates, to show how it is possible to avoid them and to compare the basic approaches are the other goals of the paper.

The paper is organised as follows. The problem is defined in Section 2. Section 3 discusses the constrained estimation in the probability density framework. Section 4 describes constrained fusion approaches. An example is given in Section 5 and the results are summarised in Section 6.

2 Problem statement

In this section, the constrained estimation problem is defined in accord with literature and its formulation by probability densities is proposed.

The linear multisensor system is *described* by

$$\mathbf{x}_{k+1} = \mathbf{F}_k\mathbf{x}_k + \mathbf{G}_k\mathbf{w}_k, \tag{1}$$

$$\mathbf{z}_k^{(j)} = \mathbf{H}_k^{(j)}\mathbf{x}_k + \mathbf{v}_k^{(j)}, \quad j = 1, \ldots, N, \tag{2}$$

where $\mathbf{F}_k \in \mathrm{R}^{n_x \times n_x}$, $\mathbf{H}_k^{(j)} \in \mathrm{R}^{n_z^{(j)} \times n_x}$, and $\mathbf{G}_k \in \mathrm{R}^{n_x \times n_w}$ are known matrices, k is time instant, $\mathbf{x}_k \in \mathrm{R}^{n_x}$ is the state and $\mathbf{z}_k^{(j)} \in \mathrm{R}^{n_z^{(j)}}$ is the local measurement coming from j-th sensor. The variables $\mathbf{w}_k \in \mathrm{R}^{n_w}$ and $\mathbf{v}_k^{(j)} \in \mathrm{R}^{n_z^{(j)}}$ represent uncertainties in the state and measurement equations and are modelled by white Gaussian noises with zero mean and with covariance matrices $\mathbf{Q}_k$, $\mathbf{R}_k^{(jj)}$, respectively. The processes $\{\mathbf{v}_k^{(j)}\}$ are independent of the process $\{\mathbf{w}_k\}$ and all of them are independent on the initial state described by the Gaussian pdf $p(\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_0 : \bar{\mathbf{x}}_0, \mathbf{P}_0)$. The measurement error processes $\{\mathbf{v}_k^{(j)}\}$ can be generally mutually dependent, with cross-correlations , $\mathbf{R}_k^{(ij)} = \mathrm{E}(\mathbf{v}_k^{(i)} \mathbf{v}_k^{(j)\mathrm{T}})$, but there are often assumed to be independent, $\mathbf{R}_k^{(ij)} = 0$ for $i \neq j$.

Moreover, it is known that the system state obeys equality or inequality constraints given by

$$c_e(\mathbf{x}_k) = 0, \tag{3}$$
$$c_n(\mathbf{x}_k) \leq 0. \tag{4}$$

There is much confusion over the meaning of the constraints that sources from overlooking the fact that the equations (1), (2) describe a model and not the true system. The system state is constrained by (3) or (4) and is only approximated by (1) and (2).

Now, the problem can be formulated by probability densities. The true system state denoted by † follows the unknown transition density

$$p(\mathbf{x}_{k+1}^\dagger | \mathbf{x}_k^\dagger) \tag{5}$$

with the unknown initial condition $p(\mathbf{x}_0^\dagger)$, but the set $\mathcal{C}_k$ of admissible system states is known for each time-step k,

$$\mathcal{C}_k = \{\forall \mathbf{x}_k^\dagger : p(\mathbf{x}_k^\dagger) \neq 0\}, \tag{6}$$

and some *approximation* of the system dynamics is available,

$$p(\mathbf{x}_{k+1} | \mathbf{x}_k), p(\mathbf{x}_0). \tag{7}$$

The true measurement probability density is given by

$$p(\mathbf{z}_k^{(1)\dagger}, \ldots, \mathbf{z}_k^{(N)\dagger} | \mathbf{x}_k^\dagger). \tag{8}$$

The measurement process can be desribed exactly. The model usually differs in the condition, it is assumed that the density exists also for $\mathbf{x}_k \notin \mathcal{C}_k$,

$$p(\mathbf{z}_k^{(1)}, \ldots, \mathbf{z}_k^{(N)} | \mathbf{x}_k). \tag{9}$$

Note that if the system is constrained by equality, (3), the sets $\mathcal{C}_k$ (6) are zero-measure and the densities (5) are improper.

Let each sensor have its estimator, i.e. there exist N conditional densities $p(\mathbf{x}|\mathcal{Z}^{(j)})$, where $\mathcal{Z}^{(j)}$ denotes the information which was used to compute the conditional density and may consist of measurements and constraints. The densities are often approximated by their means $\hat{\mathbf{x}}^{(j)}$ and covariance matrices $P^{(j)}$. The estimators are connected with some others by data link. It is assumed that data measured at other sensor nodes can not be processed directly, e.g. due to the unknown measurement equation of the respective sensors, or the communication of the measurements is ineffective. So it is assumed that only the estimates are communicated. The goal of the fusion is to combine local estimates which may but need not to obey the known state constraint.

3 Techniques for enforcing constraints

In this section, the existing approaches to the constrained estimation will be discussed in the probability density framework.

Assuming the model (7), (9) of the system (5), (8) is precise, the filtering densities obeys the constraint, i.e. they are zero for all $\mathbf{x}_k \notin \mathcal{C}_k$, and are given by

$$p(\mathbf{x}_k | \mathbf{z}_k, \mathcal{Z}_{k-1}) \propto p(\mathbf{z}_k | \mathbf{x}_k) p(\mathbf{x}_k | \mathcal{Z}_{k-1}), \tag{10}$$

where the indices $^{(j)}$ are omitted and $\propto$ means proportional to, and the predictive densities

$$p(\mathbf{x}_{k+1} | \mathcal{Z}_k) = \int_R p(\mathbf{x}_{k+1} | \mathbf{x}_k) p(\mathbf{x}_k | \mathcal{Z}_k) \mathrm{d}\mathbf{x}_k \tag{11}$$

obeys the constraint as well. But the model is always an approximation of the true system and thus the estimates can violate the constraints. To cope with this problem, various approaches exist.

3.1 Modification of state model

The simplest idea is to change the model by model projection or reduction

$$\mathrm{C}_S : p(\mathbf{x}_{k+1} | \mathbf{x}_k) \to p_{\mathcal{C}}(\mathbf{x}_{k+1} | \mathbf{x}_k) \tag{12}$$

in such a way that the transformed densities obey the constraint,

$$p_{\mathcal{C}}(\mathbf{x}_{k+1} | \mathbf{x}_k) \neq 0 \quad \Rightarrow \quad \mathbf{x}_{k+1} \in \mathcal{C}_{k+1}, \tag{13}$$
$$p(\mathbf{x}_0) \neq 0 \quad \Rightarrow \quad \mathbf{x}_0 \in \mathcal{C}_0. \tag{14}$$

This is hardly achievable in the case of inequality constraints and can lead to the loss of physical interpretation of the state if the state dimension is reduced in the equality constraint case. But it solves the problem once and for all.

3.2 Modification of measurement model

Another solution is to consider the model to be accurate and use the constraints as additive information which removes the inadmissible states. In the inquality constraint case, the pdf truncation represent this approach. The information given by the set $\mathcal{C}_k$ will be represented by conditional density $p(\mathbf{x}_k|\mathbf{c}_k)$ where $\mathbf{c}_k$ can be treated as a fictive measurement. The fusion of the unconstrained estimate with the constraining information is given by

$$p(\mathbf{x}_k|\mathcal{Z}_k \cup \mathbf{c}_k) \propto p(\mathbf{x}_k|\mathcal{Z}_k)p(\mathbf{x}_k|\mathbf{c}_k) \tag{15}$$

where no common information prior to fusion is assumed, i.e. $p(\mathbf{x}_k|\mathcal{Z}_k \cap \mathbf{c}_k) \propto 1$. The constraining density is considered to be uniform over the $\mathcal{C}_k$,

$$p(\mathbf{x}_k|\mathbf{c}_k) \propto \mathrm{I}_{\mathcal{C}_k}(\mathbf{x}_k), \tag{16}$$

where $\mathrm{I}_{\mathcal{C}_k}(\mathbf{x}_k)$ is the indicator function, $\mathrm{I}_{\mathcal{C}_k}(\mathbf{x}_k) = 1$ if $\mathbf{x}_k \in \mathcal{C}_k$ and zero otherwise.

The equality constraint case is represented by the perfect measurement approach. Let the $\mathcal{C}_k$ in (6) be given by the solution to $c_k(\mathbf{x}_k) = 0$. The value $\mathbf{c}_k = c_k(\mathbf{x}_k)$ is treated as a measurement. It must hold $p(\mathbf{c}_k|\mathbf{x}_k^\dagger) = \delta(\mathbf{c}_k)$ and $p(\mathbf{c}_k = 0|\mathbf{x}_k) = 0$, $\mathbf{x}_k \notin \mathcal{C}_k$. Obeying the constraint means to have a perfect measurement $\mathbf{c}_k = 0$ for all k. Then the estimated density is filtered by $p(\mathbf{c}_k|\mathbf{x}_k) = \delta(\mathbf{c}_k - c_k(\mathbf{x}_k))$ similarly to (10). The result can be also written as

$$p(\mathbf{x}_k|\mathcal{Z}_k, \mathbf{c}_k) \propto \mathrm{I}_{\mathcal{C}_k}(\mathbf{x}_k)p(\mathbf{x}_k|\mathcal{Z}_k). \tag{17}$$

This equation differs from the (15) in the set $\mathcal{C}_k$, in (17), the $\mathcal{C}_k$ is measure-zero set. In (15), the $\mathbf{c}_k$ is a fictious measurement in posterior density introduced for notational purpose, whereas in (17), the $\mathbf{c}_k$ is treated as a usual measurement with zero noise in likelihood function. Note that $p(\mathbf{x}_k|\mathcal{Z}_k, \mathbf{c}_k)$ is improper density.

A numerical solution to the perfect measurement approach often introduces so called soft constraints, the density of the perfect measurement $p(\mathbf{c}_k|\mathbf{x}_k) = \delta(\mathbf{c}_k - c_k(\mathbf{x}_k))$ is approximated by $p(\mathbf{c}_k|\mathbf{x}_k) \approx \mathcal{N}(\mathbf{c}_k : c_k(\mathbf{x}_k), \mathbf{R})$, $\mathbf{R} \approx 0$ and the resulting equation corresponds to the standard filtering one,

$$p(\mathbf{x}_k|\mathcal{Z}_k, \mathbf{c}_k) \propto p(\mathbf{c}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathcal{Z}_k), \tag{18}$$

where $\mathbf{c}_k = 0$ is always measured.

3.3 Projection approach

Another approach to cope with the violating of the constraints is to admit that the model is only an approximation of the true system. Then the estimates are approximations too. Thus the conflict with the constraints is the consequence of the uncertainty in the model and the conflicting states should not be rejected as in the previous approaches but projected on the constraint. Rejection removes the ignorance of the inadmissibility of the state whereas the projection decreases the uncertainty in the density function.

The estimate projection approach transforms the unconstrained density by C_k,

$$\mathrm{C}_k : p(\mathbf{x}_k|\mathcal{Z}_k) \rightarrow p(\mathbf{x}_k|\mathcal{Z}_k, \mathbf{c}_k), \tag{19}$$

so that $p(\mathbf{x}_k|\mathcal{Z}_k, \mathbf{c}_k) \neq 0 \Rightarrow \mathbf{x}_k \in \mathcal{C}_k$.

In this paper, the projection approach will be used.

4 Fusion with constraints

This section discusses the estimate fusion in a multi-sensor system with state constraints. Subsections 4.1 describes a fusion method. Subsection 4.2 shows some problems connected with the fusion of constrained estimates. A solution is proposed in Subsection 4.3 by using the unscented transformation for the estimate projection. Finally, Subsection 4.4 proposes basis approaches to constrained fusion.

4.1 Fusion of estimates

The key issue in the estimate fusion is the common prior information or let say the dependence of the estimates. Strong assumptions are required even in the unconstrained estimation to be able to compute it. The estimate projection or pdf truncation blur the meaning of the common prior information so it will be assumed that it is not available. The Covariance Intersection algorithm, see [6, 7], can be used to fuse the estimates. Generalisations of the algorithm exist, see [8, 9] for example, but their practical implementation is difficult.

The basic Covariance Intersection algorithm fuses two consistent estimates represented by their mean and covariance, $\{\hat{\mathbf{x}}_1, \mathbf{P}_1\}$, $\{\hat{\mathbf{x}}_2, \mathbf{P}_2\}$ to a consistent estimate $\{\hat{\mathbf{x}}, \mathbf{P}\}$ irrespective of the cross-covariance $\mathbf{P}_{12} = \mathrm{E}[(\mathbf{x} - \hat{\mathbf{x}}_1)(\mathbf{x} - \hat{\mathbf{x}}_2)^\mathrm{T}]$ of the estimates. The consistency is defined by

$$\mathbf{P} - \mathrm{E}[(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^\mathrm{T}] \geq 0, \tag{20}$$

where $\mathbf{x}$ represents the true state and ≥ 0 means positive semidefinite. The fusion is given by

$$\begin{aligned} \mathbf{P}^{-1}\hat{\mathbf{x}} &= \omega \mathbf{P}_1^{-1}\hat{\mathbf{x}}_1 + (1-\omega)\mathbf{P}_2^{-1}\hat{\mathbf{x}}_2, && (21) \\ \mathbf{P}^{-1} &= \omega \mathbf{P}_1^{-1} + (1-\omega)\mathbf{P}_2^{-1}, && (22) \end{aligned}$$

where $\omega \in [0, 1]$ is a free weighting constant that can be chosen in order to minimise various criteria, usually the uncertainty of the fused estimate represented by the determinant of the fused covariance matrix,

$$\omega^* = \arg\min_{\omega \in [0,1]}(\det \mathbf{P}). \tag{23}$$

4.2 Some aspects of constrained fusion algorithm

Now, a critical issue in the fusion of equality constrained estimates will be discussed. The fusion rule (21), (22) uses matrix inverses (the scalar weigts ω,

$1-\omega$ will be omitted here). As discussed in the section 3, the equality constrained densities are improper. A linear approximation can result in a singular covariance matrix, so one can try to use pseudoinverses. The Moore-Penrose pseudoinverse will be denoted by $^{-1_{MP}}$. If both estimates are constrained in the same direction, $m \gg 0$, $n \gg 0$,

$$\left(\begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix}^{-1_{MP}} + \begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix}^{-1_{MP}}\right)^{-1_{MP}} = \begin{bmatrix} \frac{1}{1/m+1/n} & 0 \\ 0 & 0 \end{bmatrix}, \tag{24}$$

no problem occurs, the information is summed properly. However, if the constraint is nonlinear, using of the linear approximation of the projected covariance matrix (28) is tricky. Applying inverses in (22) to matrices that are almost singular, $a \gg b$, $b \to 0$, $c \ll d$, $c \to 0$, leads to expected result,

$$\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^{-1} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}^{-1}\right)^{-1} \doteq \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{25}$$

whereas the use of pseudoinverses on singular matrices constrained in different directions spoils the fusion,

$$\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}^{-1_{MP}} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}^{-1_{MP}}\right)^{-1_{MP}} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \gg \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{26}$$

The explication is simple, the pseudoinverse maps certainty to zero information. If a zero information is added in the correponding direction as in (24), the outer pseudoinverse maps the fused information back to certainty. If the uncertainty is close to zero, the information is nearly infinite, that does not change if a finite number is added, and the outer inverse maps the infinity to zero properly, like in (25). Adding a nonzero to zero instead of infinity is fatal, as shown in (26).

4.3 Unscented transformation for estimate projection

The estimate projection (19) shifts the states that do not obey the constraint to the constraint domain,

$$\pi(\mathbf{x}_k) : \mathbf{x}_k \to \mathbf{x}_k^\dagger, \tag{27}$$

The equality constrained density can be computed for Gaussian densities and linear constraint. In such case, the mean of the constrained density $\hat{\mathbf{x}}_k^\dagger$ is the projection of the mean of the unconstrained density $\hat{\mathbf{x}}_k$ and the constrained covariance matrix $\mathbf{P}_k^\dagger$ is given by

$$\mathbf{P}_k^\dagger = \nabla\pi(\hat{\mathbf{x}}_k)\mathbf{P}_k\nabla\pi(\hat{\mathbf{x}}_k)^{\mathrm{T}}, \tag{28}$$

where $\mathbf{P}_k$ denotes the unconstrained covariance matrix and $\nabla\pi(\hat{\mathbf{x}}_k)$ the Jacobian matrix of the projection $\pi(\mathbf{x}_k)$ evaluated in $\hat{\mathbf{x}}_k$.

In general cases, the projection (19) can not be solved analytically. A global approach like a particle filtering can be used or the density can be approximated by its mean and covariance. The problem is that the projection of the mean need not to be the mean of the projected density and mainly that the relation (28) becomes approximative that can lead to (26). Moreover, a derivative of the projection has to be computed in (28). This can be very demanding if the projection (27) is computed by a numerical method like in [10]. The unscented transformation [11] can solve the above mentioned issues.

The basic idea of the unscented transformation is given as follows. The estimate $\hat{\mathbf{x}}, \mathbf{P}$ is substitued by a set of points $\mathcal{X}$ with corresponding weights $\mathcal{W}$ according to

$$\mathcal{X}_o = \hat{\mathbf{x}}, \qquad \mathcal{W}_0 = \frac{\kappa}{n_x+\kappa}, \tag{29}$$

$$\mathcal{X}_i = \hat{\mathbf{x}} + \left(\sqrt{(n_x+\kappa)\mathbf{P}}\right)_i, \qquad \mathcal{W}_i = \frac{1}{2(n_x+\kappa)}, \tag{30}$$

$$\mathcal{X}_j = \hat{\mathbf{x}} + \left(\sqrt{(n_x+\kappa)\mathbf{P}}\right)_{j-n_x}, \qquad \mathcal{W}_j = \frac{1}{2(n_x+\kappa)}, \tag{31}$$

$i = 1, \ldots, n_x$ $j = n_x+1, \ldots, 2n_x$, where $\left(\sqrt{(n_x+\kappa)\mathbf{P}}\right)_i$ is the i-th column of the matrix $\sqrt{(n_x+\kappa)\mathbf{P}}$ with the square root obtained by the SVD algorithm, $\mathbf{P} = \mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$, $\sqrt{\mathbf{P}} = \mathbf{U}\sqrt{\Sigma}$, n_x is the state dimension and κ is a tuning constant. The states $\mathcal{X}$ are projected, $\mathcal{X}^p = \pi(\mathcal{X})$, and the projected density is approximated by mean $\hat{\mathbf{x}}^p$ and covariance $\mathbf{P}^p$,

$$\hat{\mathbf{x}}^p = \sum_{i=0}^{2n_x} \mathcal{W}_i\mathcal{X}_i^p, \tag{32}$$

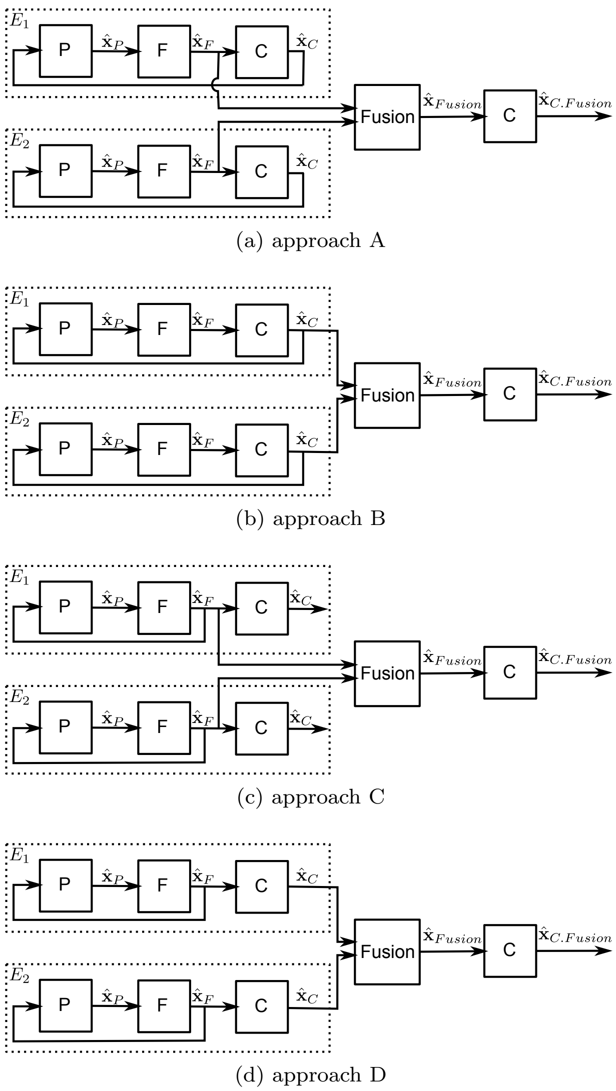
$$\mathbf{P}^p = \sum_{i=0}^{2n_x} \mathcal{W}_i(\mathcal{X}_i^p - \hat{\mathbf{x}}^p)(\mathcal{X}_i^p - \hat{\mathbf{x}}^p)^{\mathrm{T}}. \tag{33}$$

Remember that the constrained mean need not to obey the constraint, if the constraint is nonlinear. Various approaches have been proposed in [12] to deal with this issue. At least, enforcing the constrait improves the covariance.

4.4 Constraint and fusion in estimator architecture

There are many possibilities how to design a multisensor estimation with constraints. The constraints can be applied during the estimation process to the filtering and/or predictive densities or they can be applied to an unconstrained estimator outside the prediction-filtering loop. The fusion can be done in a fusion centre without any feedback to the local estimators, with communication of the fused estimate to the local estimators or decentrally. The predictive, filtering or their constrained densities can be communicated and fused.

In this paper, it will be assumed that only the filtering or constrained filtering estimates are worth to be



(a) approach A

(b) approach B

(c) approach C

(d) approach D

Figure 1: Diagrams of basic approaches a, b, c, d to constrained fusion for two local estimators $E1$, $E2$; P=prediction, F=filtering, C=constraining

communicated, because the prediction adds no information to them.

Figure 1 shows four basic approaches to the constrained fusion that will be inspected in this article. The estimates are fused at a fusion centre without feedback to the local estimators. In the approaches A and B, the local estimators use the projection (19), that constraints the filtering estimate, in a loop. In the approaches C and D, the estimators are classical and the projection (19) is done outside the loop. An alternative exists in the approaches B and D; instead of the constrained estimates, the filtering estimates can be sent to the fusion centre (as in the approaches A and C) and the constraining step can be done at the fusion centre prior the fusion. After the fusion, even if the local estimates are constrained, the fused estimate is projected

Table 1: Comparison of constrained fusion approaches that are shown at Figure 1

	fusion of constrained estimates	fusion of unconstrained estimates
constraint in the loop	A	B
constraint outside the loop	C	D

onto the constraint. Table 1 contains an outlook on the approaches.

For the linear Gaussian system (1), (2), the filtering (10) and the predictive (11) densities are Gaussian, where the means and covariances are given by the Kalman filter,

$$\hat{\mathbf{x}}_{k,F} = \hat{\mathbf{x}}_{k,P} + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}_k\hat{\mathbf{x}}_{k,P}), \tag{34}$$

$$\mathbf{K}_k = \mathbf{P}_{k,P}\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_{k,P}\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}, \tag{35}$$

$$\mathbf{P}_{k,F} = \mathbf{P}_{k,P} - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_{k,P}, \tag{36}$$

$$\hat{\mathbf{x}}_{k+1,P} = \mathbf{F}_k\hat{\mathbf{x}}_{k,*}, \tag{37}$$

$$\mathbf{P}_{k+1,P} = \mathbf{F}_k\mathbf{P}_{k,*}\mathbf{F}_k^{\mathrm{T}} + \mathbf{G}_k\mathbf{Q}_k\mathbf{G}_k^{\mathrm{T}}, \tag{38}$$

where the indices $^{(j)}$ and $^{(jj)}$ are omitted for notational simplicity and $*$ can be F or C according to the type of constraint application. The constraining step via the estimate projection is described in the subsection 4.3, $(\hat{\mathbf{x}}_{k,F}, \mathbf{P}_{k,F}) \rightarrow (\hat{\mathbf{x}}_{k,C}, \mathbf{P}_{k,C})$. The fusion is given by (21)-(23).

5 Numerical Example

As an example of the constrained estimation, a ground vehicle tracking is considered. The vehicle is moving along a circular road, where the centre and the radius of the turn are known, with a constant angular velocity. The state is given by the positions x, y, and velocities v_x, v_y, in the x-y coordinates, $\mathbf{x} = [x, v_x, y, v_y]^{\mathrm{T}}$.

The model of nearly constatnt velocity is used. The model dynamics is given by (1) where

$$\mathbf{F}_k = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{G}_k = \begin{bmatrix} \frac{1}{2}T^2 & 0 \\ T & 0 \\ 0 & \frac{1}{2}T^2 \\ 0 & T \end{bmatrix}, \tag{39}$$

the sampling period T is $1s$ and the noise covariance matrices $\mathbf{Q}_k = \mathrm{diag}(4m^2/s^4, 4m^2/s^4)$ is chosen.

Two sensors measure the position of the vehicle, $N = 2$. The measurements are functions of the real vehicle position,

$$\mathbf{z}_k^{(j)\dagger} = \mathbf{H}_k^{(j)\dagger}\mathbf{x}_k^{\dagger} + \mathbf{v}_k^{(j)\dagger}, \quad j = 1, \ldots, 2, \tag{40}$$

where the measurement matrices are given as

$$\mathbf{H}_k^{(j)\dagger} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{41}$$

the measurement noises $\mathbf{v}_k^{(j)\dagger}$ are independent and their covariances are $\mathbf{R}_k^{(jj)\dagger} = \mathrm{diag}(50m^2, 50m^2)$. The sensor model is given by (2) where $\mathbf{H}_k^{(j)} = \mathbf{H}_k^{(j)\dagger}$ and $\mathbf{R}_k^{(jj)} = \mathbf{R}_k^{(jj)\dagger}$. The true measurement value $\mathbf{z}_k^{\dagger}$ is used in the filtering steps (34)-(36), $\mathbf{z}_k = \mathbf{z}_k^{\dagger}$.

In the following part, the time indices k will be omitted for notational simplicity. Let the centre of the turn lie in the origin of the x-y coordinates and the turn radius to be $R = 70m$. The equality constraint (3) is given by

$$\begin{aligned} x^2 + y^2 - 70^2 &= 0, & (42) \\ xv_x + yv_y &= 0. & (43) \end{aligned}$$

The vehicle speed is $12m/s$ and the initial condition is given by $\mathbf{x}_0 = \mathbf{x}_0^{\dagger} = [70m, 0m/s, 0m, 5m/s]^{\mathrm{T}}$, $\mathbf{P}_0 = \mathrm{diag}(0.5m^2, 1m^2/s^2, 5m^2, 1m^2/s^2)$.

The projection function (27) projects the position of the vehicle onto the turn and the velocity to the tangent of the turn [10] and is given by

$$\pi(\mathbf{x}) = \begin{bmatrix} \frac{Rx}{\sqrt{x^2+y^2}} & \frac{v_x y^2 - v_y xy}{x^2+y^2} & \frac{Ry}{\sqrt{x^2+y^2}} & \frac{-v_x xy + v_y x^2}{x^2+y^2} \end{bmatrix}^{\mathrm{T}}. \tag{44}$$

The estimates are projected by the unscented transformation, (29)-(33) where $\mathcal{X}^p = \pi(\mathcal{X})$ is given by (44).

The approaches A, B, C and D shown in Figure 1 are compared and corresponding methods are described in Section 4. The position and velocity error,

$$\begin{aligned} \varepsilon_{position}(\hat{\mathbf{x}}) &= \sqrt{(x-\hat{x})^2 + (y-\hat{y})^2}, & (45) \\ \varepsilon_{velocity}(\hat{\mathbf{x}}) &= \sqrt{(v_x-\hat{v}_x)^2 + (v_y-\hat{v}_y)^2}, & (46) \end{aligned}$$

will be inspected. The simulation runs for 500 time steps, $k = 1, \ldots, 500$. The mean values of the errors during the simulation are given by

$$\bar{\varepsilon}(\hat{\mathbf{x}}) = \frac{1}{500} \sum_{k=1}^{500} \varepsilon(\hat{\mathbf{x}}_k). \tag{47}$$

Table 2 compares the means of the fused and constrained fused estimate errors. The fusion of unconstrained estimates with the constraint outside the loop given by the approach C is the standard fusion, no constraining step is used to obtain this estimate. The approaches A, B and D, that use constraining steps prior the fusion step, give better results than the standard unconstrained fusion, $\bar{\varepsilon}(\hat{\mathbf{x}}_{Fusion})\{A,B,D\} < \bar{\varepsilon}(\hat{\mathbf{x}}_{Fusion})\{C\}$, for both the position and velocity, so both the indices are omitted.

The approaches B and D fuse constrained estimates, so the constraining of the fusion does not improve the fusion notably, $\bar{\varepsilon}(\hat{\mathbf{x}}_{C.Fusion})\{B,D\} \doteq \bar{\varepsilon}(\hat{\mathbf{x}}_{Fusion})\{B,D\}$. Fusing the constrained estimates $\hat{\mathbf{x}}_C$ is better that fusing the unconstrained estimates $\hat{\mathbf{x}}_F$, $\bar{\varepsilon}\{B\} < \bar{\varepsilon}\{A\}$, $\bar{\varepsilon}\{D\} < \bar{\varepsilon}\{C\}$, for both the unconstrained and constrained fusion estimate $\hat{\mathbf{x}}_{Fusion}$ and $\hat{\mathbf{x}}_{C.Fusion}$. However, there may arise serious problems. If the constrained estimates have appropriate singular covariance matrices as in (24), the determinant of the fused estimate is zero and a change of the fusion criterion (23) is needed. Further, the fusion of inappropriately approximated covariances (26) chooses one constrained estimate to be the fused estimate, but not necessarily the better one. These problems were avoided by the use of the unscented transformation for the estimate projection (44).

Table 2: Mean position and velocity errors after the fusion of local estimates, the standard unconstrained fusion is given in italics in the column C

approach	A	B	C	D
$\bar{\varepsilon}_{position}(\hat{\mathbf{x}}_{Fusion})$	2.75	2.40	*6.88*	2.91
$\bar{\varepsilon}_{velocity}(\hat{\mathbf{x}}_{Fusion})$	2.27	1.01	*4.86*	1.20
$\bar{\varepsilon}_{position}(\hat{\mathbf{x}}_{C.Fusion})$	2.49	2.39	3.53	2.91
$\bar{\varepsilon}_{velocity}(\hat{\mathbf{x}}_{C.Fusion})$	1.04	1.01	1.43	1.20

Table 3: Mean position and velocity errors at local estimators E_1, E_2; symbols given by Figure 1, the standard unconstrained estimation are given in italics in the column C,D

	$E_1\{A,B\}$	$E_2\{A,B\}$	$E_1\{C,D\}$	$E_2\{C,D\}$
$\bar{\varepsilon}_{position}(\hat{\mathbf{x}}_P)$	4.34	4.67	*12.78*	*12.07*
$\bar{\varepsilon}_{velocity}(\hat{\mathbf{x}}_P)$	2.41	2.41	*6.90*	*6.79*
$\bar{\varepsilon}_{position}(\hat{\mathbf{x}}_F)$	3.52	3.77	*7.65*	*7.19*
$\bar{\varepsilon}_{velocity}(\hat{\mathbf{x}}_F)$	2.45	2.43	*5.00*	*4.91*
$\bar{\varepsilon}_{position}(\hat{\mathbf{x}}_C)$	3.29	3.56	3.76	3.76
$\bar{\varepsilon}_{velocity}(\hat{\mathbf{x}}_C)$	1.34	1.43	1.51	1.52

Table 3 focuses on the quality of the estimates prior the fusion. The local estimates, and consequently their errors, at local estimators E_1, E_2, are identical for the approaches A and B and for C and D, that can be deduced from the Table 1. The errors of predictive, filtered and constrained filtering estimates are compared. The corresponding errors of the first and second estimator are almost the same, $\bar{\varepsilon}[E_1] \approx \bar{\varepsilon}[E_2]$, because the sensors have the same measurement error covariance matrix. The local estimators with the constraint in the loop (approaches A and B) give lower errors than those with the constraint outside the loop (approaches C and D), $\bar{\varepsilon}\{A,B\} < \bar{\varepsilon}\{C,D\}$. The predictive and filtering es-

Table 4: Number of constraining steps needed to compute the constrained fusion estimate $\hat{\mathbf{x}}_{k,C.Fusion}$, $N=$ number of estimators, $k=$ time instant

approach	A	B	C	D
number of constraining steps	$kN+1$	$(k+1)N+1$	1	$N+1$

timate errors $\bar{\varepsilon}(\hat{\mathbf{x}}_P)$, $\bar{\varepsilon}(\hat{\mathbf{x}}_F)$, are notably lower, but the constrained estimates $\bar{\varepsilon}(\hat{\mathbf{x}}_C)$ are relatively close.

Table 4 shows the number of constraining steps needed to compute the constrained fusion estimate $\hat{\mathbf{x}}_{C.Fusion}$ for the discussed approaches. As can be seen from the comparison with the last two rows of the table 2, the mean error of the constrained fusion estimate $\bar{\varepsilon}(\hat{\mathbf{x}}_{C.Fusion})$ decreasses with this number.

It has been shown that the choice of an approach influences the computation demands and the results. There exist other significant aspects, especially the divergence between the model and the system. If the model is very close to the system, the constraint can not bring much improvement. The number of constrained states or constraints influences the results too. For example, only the position constraint (42) can be applied, though the physical laws induce some constraint of the velocity. It is possible that using incomplete constraint once gives better results than using it multiple times, i.e. using the constraint outside the loop can be better than using it in the loop. Further, the choice of the projection influences the results significantly, however, if the projection is reasonable, i.e. if the distance of the unconstrained and the projected estimate is low, good results can be expected.

6 Summary

The multisensor constrained estimators were designed and it was shown that the constrained fusion provides better results than the unconstrained fusion. Four basic approaches for constrained estimate fusion were developed. Fusion of constrained or unconstrained estimates with the constraint in the loop or oustside the loop was discussed. Linearisation of non-linear estimate projection may lead to spurious fusion results and therefore the unscented transformation for the projection was used.

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